

Random assignment with multi-unit demands

Haris Aziz

Received: date / Accepted: date

Abstract We consider the multi-unit random assignment problem in which agents express preferences over objects and objects are allocated to agents randomly based on the preferences. The most well-established preference relation to compare random allocations of objects is stochastic dominance (*SD*) which also leads to corresponding notions of envy-freeness, efficiency, and weak strategyproofness. We show that there exists no rule that is anonymous, neutral, efficient and weak strategyproof. For single-unit random assignment, we show that there exists no rule that is anonymous, neutral, efficient and weak group-strategyproof. We then study a generalization of the *PS* (probabilistic serial) rule called multi-unit-eating *PS* and prove that multi-unit-eating *PS* satisfies envy-freeness, weak strategyproofness, and unanimity.

Keywords Fair division · probabilistic serial rule · strategyproofness · Pareto optimality

JEL Classification: C70 · D61 · D71

1 Introduction

In the assignment problem, agents express linear preferences over objects and an object is assigned to each agent keeping in view the agents' preferences. The problem models one of the most fundamental setting in computer science and economics with numerous applications (Gärdenfors, 1973; Wilson, 1977; Young, 1995; Svensson, 1994, 1999; Bouveret et al., 2010; Abraham et al., 2005). Depending on the application setting, the objects could be car-park spaces, dormitory rooms, replacement kidneys, school seats, etc. The assignment problem is also referred to as

Haris Aziz
NICTA and UNSW 2033 Sydney, Australia
Tel.: +61-2-8306 0490
Fax: +61-2-8306 0405
E-mail: haris.aziz@nicta.com.au

house allocation (Abraham et al., 2005; Abdulkadiroğlu and Sönmez, 1999). If the outcome of the assignment problem is *deterministic* then it can be inherently unfair. Take the example of two agents having identical preferences over two objects. Then any reasonable notion of fairness demands that both agents have equal right to each of the two objects. Since randomization is one of the oldest tools to achieve fairness, we consider the *random assignment problem* (Hylland and Zeckhauser, 1979; Young, 1995; Bogomolnaia and Moulin, 2001; Katta and Sethuraman, 2006; Guo and Conitzer, 2010; Bhalgat et al., 2011; Budish et al., 2013) in which objects are allocated randomly to agents according to their preferences. The outcome is a random assignment which specifies the probability of each object being allocated to each of the agents. In contrast to some of the earlier work on random assignment, we focus on the random assignment problem in which there can be more objects than the number of agents (Kojima, 2009).

When agents express ordinal preferences over objects but the outcomes are fractional or randomized allocations, then there is a need to use *lottery extensions* to extend preferences over objects to preferences over random allocations. In random settings, the most established preference relation between random allocations is *stochastic dominance* (*SD*). *SD* requires that one random allocation is preferred to another one if and only if the former first-order stochastically dominates the latter. This relation is especially important because one random allocation stochastically dominates another one if and only if the former yields at least as much expected utility as the latter for any von-Neumann-Morgenstern (vNM) utility representation consistent with the ordinal preferences (Aziz et al., 2013c). The *SD* relation can be used to define corresponding notions of envy-free, efficiency, and strategyproofness (Bogomolnaia and Moulin, 2001; Katta and Sethuraman, 2006). In this paper, we check which levels of fairness, efficiency, and strategyproofness can be satisfied simultaneously.

For the random assignment problem without multi-unit demands, the most common and well-known way to assign objects is *random priority* (*RP*) in which a permutation of agents is chosen uniformly at random and agents successively take their most preferred available object (Abdulkadiroğlu and Sönmez, 1998; Bogomolnaia and Moulin, 2001; Crès and Moulin, 2001). Although *RP* is strategyproof and results in a Pareto optimal assignment, Bogomolnaia and Moulin (2001) in a remarkable paper showed that *RP* does not satisfy the stronger efficiency notion of stochastic dominance (*SD*) efficiency and also a fairness concept called *SD*-envy-freeness.¹ Furthermore, they presented an elegant algorithm called *PS* (*probabilistic serial*) that is not only *SD*-efficient and *SD*-envy-free but also satisfies weak *SD*-strategyproofness. In *PS*, agents ‘eat’ the most favoured available object at the same rate until all the objects are consumed. The fraction of object consumed by an agent is the probability of the agent getting that object.²

Since its inception (Bogomolnaia and Moulin, 2001), *PS* has received considerable attention and has been extended in a number of ways (Katta and Sethuraman,

¹ Another drawback of *RP* is that the resultant fractional allocation is #P-complete to compute (Aziz et al., 2013a).

² By the *Birkhoff-von Neumann theorem*, any fractional assignment can be represented by a convex combination over discrete assignments.

2006; Athanassoglou and Sethuraman, 2011; Yilmaz, 2009). In particular, it can be naturally extended to the more general case with multi-unit demands in which there are nc objects and $c \geq 1$ objects are allocated to each of the agents (Bogomolnaia and Moulin, 2001; Heo, 2011; Kojima, 2009). The extension does not require any modification to the specification of *PS*: agents continue eating their most preferred available object until all the objects have been consumed. Although this *one-at-a-time* extension (which we will refer to as *OPS*) still satisfies *SD*-efficiency and *SD*-envy-freeness, it is not weak *SD*-strategyproof (Kojima, 2009). Incidentally there is another extension of *PS* called the *multi-unit-eating probabilistic serial* that was briefly described by Che and Kojima (2010) but has received no attention in the literature. In multi-unit-eating *PS*, each agent tries to eat his c most preferred objects that are still available at a uniform speed until all objects have been consumed. We show that multi-unit-eating *PS* satisfies desirable properties: it is weak *SD*-strategyproof, *SD*-envy-free, and unanimous.

We point out that the problem of discrete assignment with multi-unit demands has attracted considerable attention (Bouveret and Lang, 2011; Budish, 2011; Ehlers and Klaus, 2003; Hatfield, 2009; Kalinowski et al., 2013; Bouveret et al., 2010). In this paper, we focus on *random* assignments with multi-unit demands. Multi-unit demand is a natural requirement in settings such as course allocation (Budish, 2011). Moreover, we will require that each agents gets equal number of objects (Hatfield, 2009). This is a natural requirement in settings such as paper assignment to referees.

Apart from *RP* and *PS*, two other natural assignment rules are *uniform* and *priority*. In the uniform rule, each agent gets $1/n$ of each object (Chambers, 2004; Kojima, 2009). In the priority mechanism, there is a permutation of agents, and each agent in the permutation is assigned the c most preferred available objects. The priority mechanism is also referred to as *serial dictator* in the literature (Svensson, 1994, 1999). Whereas uniform does not take into account the preferences of agents and is highly inefficient, priority is highly unfair to the agents at the end of the permutation. In more recent work, Nguyen et al. (2015) proposed two mechanisms for the random assignment problem that also handle limited complementarities. Hashimoto (2013) presented a generalization of *RP* for more general settings.

Contributions We first prove that for multi-unit demands, there exists no anonymous, neutral, weak *SD*-strategyproof and *SD*-efficient random assignment rule. The statement is somewhat surprising considering that all the four axioms used in the statement are minimal requirements. Incidentally, we have not used *SD* envy-freeness that is often used to obtain characterizations or impossibility statements in the literature (Heo, 2011; Bogomolnaia and Moulin, 2001; Ehlers and Klaus, 2003; Kojima, 2009) and is a very demanding requirement. The result is then extended to random assignment *without* multi-unit demands if requiring weak *SD* group-strategyproofness instead of weak *SD* strategyproofness. Our second result carries over to the setting randomized voting in which agents express weak orders over alternatives and the outcome is a lottery over the alternatives.

We then conduct an axiomatic analysis of the multi-unit-eating *PS*. It is first highlighted that the definition of multi-unit-eating *PS* in the literature is not en-

tirely correct. A proper definition of multi-unit-eating *PS* is formulated. We show that for multi-unit demands, in contrast to *OPS*, multi-unit-eating *PS* satisfies weak *SD*-strategyproofness. We prove that multi-unit-eating *PS* satisfies *SD* envy-freeness which is one of the strongest notions of fairness. On the other hand, multi-unit-eating *PS* does not fare well in terms of efficiency. We prove that multi-unit-eating *PS* does not even satisfy ex post efficiency although it does satisfy unanimity. Therefore when we generalize *PS* for multi-unit demands, *OPS* is the right extension if the focus is on efficiency. On the other hand multi-unit-eating *PS* is the right extension, if the aim is to maintain weak *SD*-strategyproofness. The arguments for weak *SD*-strategyproofness and *SD* envy-freeness of *MPS* multi-unit-eating *PS* also simplify the proofs for *PS* for single-unit demands in (Bogomolnaia and Moulin, 2001). The study helps clarify the relative merits of different assignment rules for multi-unit demands. The relative merits of prominent random assignment rules are then summarized in Table 1 in the final section.

2 Preliminaries

Random assignment problem The model we consider is the random assignment problem which is a triple (N, O, \succsim) where N is the set of n agents $\{1, \dots, n\}$, $O = \{o_1, \dots, o_m\}$ is the set of objects, and $\succsim = (\succsim_1, \dots, \succsim_n)$ specifies strict, complete, and transitive preferences \succsim_i of agent i over O . We will assume that m is a multiple of n i.e., $m = nc$ where c is an integer. We will denote by $\mathcal{R}(O)$ as the set of all complete and transitive relations over the set of objects O .

A random assignment p is a $(n \times m)$ matrix $[p(i)(o_j)]_{1 \leq i \leq n, 1 \leq j \leq m}$ such that for all $i \in N$, and $o_j \in O$, $p(i)(o_j) \in [0, 1]$; $\sum_{i \in N} p(i)(o_j) = 1$ for all $j \in \{1, \dots, m\}$; and $\sum_{o_j \in O} p(i)(o_j) = c$ for all $i \in N$. The value $p(i)(o_j)$ represents the probability of object o_j being allocated to agent i . Each row $p(i) = (p(i)(o_1), \dots, p(i)(o_m))$ represents the allocation of agent i . The set of columns correspond to probability vectors of the objects o_1, \dots, o_m . A feasible random assignment is discrete if $p(i)(o) \in \{0, 1\}$ for all $i \in N$ and $o \in O$. A *random assignment rule* specifies for each preferences profile a random assignment. Two minimal fairness conditions for rules are *anonymity* and *neutrality*. Informally, they require that the rule should not depend on the names of the agents or objects respectively.

We define the *SD* (*stochastic dominance*) relation which is an incomplete relation that extends the preferences of the agents over objects to preferences over random allocations. Given two random assignments p and q , $p(i) \succsim_i^{SD} q(i)$ i.e., a player i *SD prefers* allocation $p(i)$ to allocation $q(i)$ if $\sum_{o_j \in \{o_k : o_k \succsim_i o\}} p(i)(o_j) \geq \sum_{o_j \in \{o_k : o_k \succsim_i o\}} q(i)(o_j)$ for all $o \in O$. Since *SD* is incomplete, it can be that two allocations $p(i)$ and $q(i)$ are *incomparable*: $p(i) \not\succsim_i^{SD} q(i)$ and $q(i) \not\succsim_i^{SD} p(i)$.

Next, we define the *DL* (*downward lexicographical*) relation which is a complete relation. Let $p(i)$ and $q(i)$ be two random allocations. Let $o \in O$ be the most preferred object such that $p(i)(o) \neq q(i)(o)$. Then, $p(i) \succ_i^{DL} q(i) \iff p(i)(o) > q(i)(o)$.

Example 1 Consider the random assignment problem for two agents $N = \{1, 2\}$ and four objects $O = \{o_1, o_2, o_3, o_4\}$ with the following preferences:

$$\begin{aligned} 1 &: o_1, o_2, o_3, o_4 \\ 2 &: o_2, o_1, o_3, o_4 \end{aligned}$$

Let us assume that agent 1 gets o_1 with probability one, and objects o_3 and o_4 with probability half. Then the random assignment can be represented by the following matrix.

$$p = \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \end{pmatrix}.$$

Note that agent 1's preference is $o_1 \succ_1 o_2 \succ_1 o_3 \succ_1 o_4$. Based on the preferences over objects, one can consider preferences over allocations: $p(1) \succ_1^{SD} p(2)$ and also $p(1) \succ_1^{DL} p(2)$.

Envy-freeness An assignment p satisfies *SD envy-freeness* if each agent (weakly) *SD* prefers his allocation to that of any other agent: $p(i) \succeq_i^{SD} p(j)$ for all $i, j \in N$. An assignment p satisfies *weak SD envy-freeness* if no agent strictly *SD* prefers someone else's allocation to his: $\neg[p(j) \succ_i^{SD} p(i)]$ for all $i, j \in N$. For fairness concepts, *SD* envy-freeness implies weak *SD*-envy-freeness (Bogomolnaia and Moulin, 2001).

Economic efficiency An assignment is *perfect* if each agents gets his most preferred c objects. An assignment p is *SD-efficient* if there exists no assignment q such that $q(i) \succeq_i^{SD} p(i)$ for all $i \in N$ and $q(i) \succ_i^{SD} p(i)$ for some $i \in N$. An assignment is *ex post efficient* if it can be represented as a probability distribution over the set of *SD*-efficient discrete assignments. Perfection implies *SD*-efficiency which implies ex post efficiency.

An assignment rule is *SD-efficient* (ex post efficient) if it always returns an *SD*-efficient (ex post efficient) assignment. An assignment rule satisfies *unanimity*, if it returns the perfect assignment if a perfect assignment exists.

SD-efficiency implies ex post efficiency which implies unanimity. The first implication was shown by (Bogomolnaia and Moulin, 2001). For the second implication, assume that an assignment does not satisfy unanimity, there exists a perfect assignment p but the mechanism returns some imperfect assignment q . The only *SD*-efficient assignment that gives c units to each agent is p . However since $q \neq p$, it cannot be achieved by a probability distribution over *SD*-efficient discrete assignments.

Strategyproofness A random assignment function f is *SD-strategyproof* if $f(\succeq_i)(i) \succeq_i^{SD} f(\succeq'_i, \succeq_{-i})(i)$ for all \succeq'_i and \succeq_{-i} . A random assignment function f is *weak SD-strategyproof* if $\neg[f(\succeq'_i, \succeq_{-i})(i) \succ_i^{SD} f(\succeq)(i)]$ for all $\succeq'_i \in \mathcal{R}(O)$ and $\succeq_{-i} \in \mathcal{R}(O)^{n-1}$. It is easy to see that *SD*-strategyproofness implies weak *SD*-strategyproofness (Bogomolnaia and Moulin, 2001). A random assignment function f is *weak SD-group-strategyproof* if there never exists an $S \subset N$ and $\succeq'_S \in \mathcal{R}(O)^{|S|}$ such that $f(\succeq'_S, \succeq_{-S})(i) \succ_i^{SD} f(\succeq)(i)$ for all $i \in S$ and $\succeq_{-S} \in \mathcal{R}(O)^{n-|S|}$.

3 General impossibilities

For the random assignment problem for which the number of objects is not more than the number of agents, there exists a rule (PS) that is anonymous, neutral, SD -efficient and weak SD -strategyproof. However when the number of objects is more than the number of agents, we get the following impossibility (Theorem 1).

Theorem 1 *For the random assignment problem with $c > 1$, there exists no anonymous, neutral, SD -efficient, and weak SD -strategyproof rule.*

Proof We consider a random assignment setting with two agents and four objects with the requirement that each agents gets two units of houses.

$$z_1: a, b, c, d$$

$$z_2: b, c, a, d$$

$$z'_1: b, a, c, d$$

$$z'_2: b, a, c, d$$

Let us compute $f(z_1, z'_2)$. By anonymity and neutrality of f

$$f(z_1, z'_2) = \begin{pmatrix} w & x & y & z \\ x & w & y & z \end{pmatrix}.$$

By SD -efficiency of f ,

$$f(z_1, z'_2) = \begin{pmatrix} 1 & 0 & y & z \\ 0 & 1 & y & z \end{pmatrix}.$$

By anonymity and neutrality of f ,

$$f(z_1, z'_2) = \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \end{pmatrix}.$$

By using similar arguments, SD -efficiency, anonymity, and neutrality of f implies that

$$f(z'_1, z_2) = \begin{pmatrix} 1 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \end{pmatrix}.$$

Now let us consider

$$f(z_1, z_2) = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}.$$

For $f(z_1, z_2)$ to be feasible,

$$x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24} \geq 0$$

$$x_{11} + x_{12} + x_{13} + x_{14} = 2$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 2$$

$$x_{11} + x_{21} = x_{12} + x_{22} = x_{13} + x_{23} = x_{14} + x_{24} = 1$$

Next, we show that if $f(\succsim_1, \succsim_2) = f(\succsim'_1, \succsim_2)$ or $f(\succsim_1, \succsim_2) = f(\succsim_1, \succsim'_2)$, then f is not weak SD -strategyproof.

If $f(\succsim_1, \succsim_2) = f(\succsim'_1, \succsim_2)$, then

$$f(\succsim_1, \succsim'_2)(2) \succ_2^{SD} f(\succsim_1, \succsim_2)(2).$$

Hence, f is not weak SD -strategyproof.

If $f(\succsim_1, \succsim_2) = f(\succsim_1, \succsim'_2)$, then

$$f(\succsim'_1, \succsim_2)(1) \succ_1^{SD} f(\succsim_1, \succsim_2)(1).$$

Hence, f is not weak SD -strategyproof.

Therefore the only way f can still be weak SD -strategyproof if both of the following conditions hold.

- $f(\succsim_1, \succsim_2)(1)$ is incomparable for 1 with $f(\succsim'_1, \succsim_2)(1)$.
- $f(\succsim_1, \succsim_2)(2)$ is incomparable for 2 with $f(\succsim_1, \succsim'_2)(2)$.

This means that the following constraints should hold.

Given that agent 2 reports \succsim_2 , agent 1 should not benefit by misreporting \succsim'_1 instead of \succsim_1 . This implies that $x_{11} + x_{12} + x_{13} > 1.5$.

Given that agent 1 reports \succsim_1 , agent 2 should not benefit by misreporting \succsim'_2 instead of \succsim_2 . This implies that $x_{22} + x_{23} + x_{21} > 1.5$.

Adding both these inequalities yields

$$x_{11} + x_{12} + x_{13} + x_{22} + x_{23} + x_{21} > 3.$$

But this is a contradiction since $x_{11} + x_{12} + x_{13} + x_{22} + x_{23} + x_{21} = (x_{11} + x_{21}) + (x_{12} + x_{22}) + (x_{13} + x_{23}) = 3$. Hence if f is SD -efficient, and anonymous, neutral, then it cannot be weak SD -strategyproof.

The same argument can be extended to arbitrary number of agents where each agent requires two objects from among o_1, \dots, o_{2n} . Each new agent $i \in \{3, \dots, n\}$ most prefers objects o_{2i-1}, o_{2i} and least prefers objects o_1, o_2, o_3, o_4 . Hence in each SD -efficient assignment each agent $i \in \{3, \dots, n\}$ is allocated o_{2i-1} and o_{2i} completely. The same arguments for the case of two agents apply to the more general case. Similarly, the same arguments can also be extended to the case where $c > 2$. One can add more objects to end of the preference lists of both agents and each agent gets a uniform fraction of these objects at the end of the preference lists. \square

Theorem 1 complements an earlier impossibility result of Kojima (2009) that states there exists no SD -efficient, SD envy-free, and weak SD -strategyproof random assignment rule for multi-unit demands. In Theorem 1, the property of SD envy-freeness is replaced by anonymity.

The proof above can be extended by cloning agents 1 and 2 to prove the following statement for the basic assignment setting with single-unit demand.

Theorem 2 *For the random assignment problem, there exists no anonymous, neutral, SD -efficient, and weak SD group-strategyproofness rule even for equal number of agents and objects.*

Proof We consider a random assignment setting with four agents and four objects. There are two agents that are of type 1 and two agents of type 2. Let the real preferences of the agents $\{1, 2\}$ of type 1 be \succsim_1 and let the real preferences of agents $\{3, 4\}$ of type 2 be \succsim_2 .

$$\begin{aligned}\succsim_1: & a, b, c, d \\ \succsim_2: & b, c, a, d \\ \succsim'_1: & b, a, c, d \\ \succsim'_2: & b, a, c, d\end{aligned}$$

Let us compute $f(\succsim_1, \succsim_1, \succsim'_2, \succsim'_2)$.

By anonymity and neutrality, we know that

$$f(\succsim_1, \succsim_1, \succsim'_2, \succsim'_2) = \begin{pmatrix} w/2 & x/2 & y/2 & z/2 \\ w/2 & x/2 & y/2 & z/2 \\ x/2 & w/2 & y/2 & z/2 \\ x/2 & w/2 & y/2 & z/2 \end{pmatrix}.$$

By *SD*-efficiency, we know that

$$f(\succsim_1, \succsim_1, \succsim'_2, \succsim'_2) = \begin{pmatrix} 1/2 & 0 & y/2 & z/2 \\ 1/2 & 0 & y/2 & z/2 \\ 0 & 1/2 & y/2 & z/2 \\ 0 & 1/2 & y/2 & z/2 \end{pmatrix}.$$

Due to anonymity and neutrality of f ,

$$f(\succsim_1, \succsim_1, \succsim'_2, \succsim'_2) = \begin{pmatrix} 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 0 & 1/2 & 1/4 & 1/4 \\ 0 & 1/2 & 1/4 & 1/4 \end{pmatrix}.$$

By using similar arguments, *SD*-efficiency, anonymity, and neutrality of f implies that

$$f(\succsim'_1, \succsim'_1, \succsim_2, \succsim_2) = \begin{pmatrix} 1/2 & 1/4 & 0 & 1/4 \\ 1/2 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/2 & 1/4 \\ 0 & 1/4 & 1/2 & 1/4 \end{pmatrix}.$$

Now let us consider

$$f(\succsim_1, \succsim_1, \succsim_2, \succsim_2) = \begin{pmatrix} x_{11}/2 & x_{12}/2 & x_{13}/2 & x_{14}/2 \\ x_{11}/2 & x_{12}/2 & x_{13}/2 & x_{14}/2 \\ x_{21}/2 & x_{22}/2 & x_{23}/2 & x_{24}/2 \\ x_{21}/2 & x_{22}/2 & x_{23}/2 & x_{24}/2 \end{pmatrix}.$$

For $f(\succsim_1, \succsim_1, \succsim_2, \succsim_2)$ to be feasible,

$$\begin{aligned}
x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24} &\geq 0 \\
x_{11} + x_{12} + x_{13} + x_{14} &= 2 \\
x_{21} + x_{22} + x_{23} + x_{24} &= 2 \\
x_{11} + x_{21} = x_{12} + x_{22} = x_{13} + x_{23} = x_{14} + x_{24} &= 1
\end{aligned}$$

Next, we show that if $f(\succsim_1, \succsim_1, \succsim_2, \succsim_2) = f(\succsim'_1, \succsim'_1, \succsim_2, \succsim_2)$ or $f(\succsim_1, \succsim_1, \succsim_2, \succsim_2) = f(\succsim_1, \succsim_1, \succsim'_2, \succsim'_2)$, then f is not weak SD group-strategyproof.

If $f(\succsim_1, \succsim_1, \succsim_2, \succsim_2) = f(\succsim'_1, \succsim'_1, \succsim_2, \succsim_2)$, then

$$f(\succsim_1, \succsim_1, \succsim'_2, \succsim'_2)(3) \succ_2^{SD} f(\succsim_1, \succsim_1, \succsim_2, \succsim_2)(3).$$

Hence, f is not weak SD group-strategyproof.

If $f(\succsim_1, \succsim_1, \succsim_2, \succsim_2) = f(\succsim_1, \succsim_1, \succsim'_2, \succsim'_2)$, then

$$f(\succsim'_1, \succsim'_1, \succsim_2, \succsim_2)(1) \succ_1^{SD} f(\succsim_1, \succsim_1, \succsim_2, \succsim_2)(1).$$

Hence, f is not weak SD group-strategyproof.

Given that agent of type \succsim_2 report \succsim_2 , then agent of type \succsim_1 should not benefit by misreporting \succsim'_1 instead of \succsim_1 . This implies that $x_{11} + x_{12} + x_{13} > 1.5$.

Given that agents of type 2 report \succsim_1 , then agents of type 2 should not benefit by misreporting \succsim'_2 instead of \succsim_2 . This implies that $x_{22} + x_{23} + x_{21} > 1.5$.

Hence,

$$x_{11} + x_{12} + x_{13} + x_{22} + x_{23} + x_{21} > 3$$

But this is a contradiction since $x_{11} + x_{12} + x_{13} + x_{22} + x_{23} + x_{21} = (x_{11} + x_{21}) + (x_{12} + x_{22}) + (x_{13} + x_{23}) = 3$. Hence if f is SD -efficient and anonymous, and neutral, then it cannot be weak SD group-strategyproof.

The same argument can be extended to arbitrary number of agents. \square

Theorem 2 (that holds for single-unit demands) complements Theorem 1 in (Kojima, 2009) that only holds for multi-unit demands. The assignment problem in which $m = n$ can be viewed as a subdomain of voting in which each alternative is a discrete assignment and preferences of an agent over assignments simply depend on his allocated object (Aziz and Stursberg, 2014). As a corollary of Theorem 2, we get that when agents may express indifference, there exists no randomized social choice rule that is anonymous, neutral, SD -efficient, and weak SD group-strategyproof. This proves a weaker version of the conjecture that there exists no randomized social choice rule that is anonymous, neutral, SD -efficient, and weak SD -strategyproof (Aziz et al., 2013b).

We now show that if one of SD -efficiency, anonymity, or weak SD -strategyproofness is dropped, then there exist rules that satisfy the other properties mentioned in the two impossibility theorems respectively even for multi-unit demands. If SD -efficiency is dropped or is replaced by ex post efficiency, then RP satisfies strategyproofness, anonymity, neutrality and ex post efficiency. If

anonymity is dropped, then the priority mechanism achieves *SD*-efficiency and group *SD*-strategyproofness. If weak *SD*-strategyproofness is dropped, then OPS satisfies the other properties. It remains open whether neutrality is necessarily required to obtain the two impossibility theorems.

4 Multi-unit-eating *PS*

In this section, we examine the properties satisfied by multi-unit-eating *PS* (*MPS*). Before we proceed, we will try to get a better understanding of how multi-unit-eating *PS* works. Che and Kojima (2010) defined multi-unit-eating *PS* as the rule in which each agent eats his c most preferred objects at speed 1 during the time interval $t \in [0, 1]$. They assumed that at each point each agent has c objects available for consumption during the running of multi-unit-eating *PS* and hence all the objects are consumed at time 1. We first show that it may be the case that less than c objects are available for consumption. Consider the illustration of multi-unit-eating *PS* in Figure 1. At time $t = 7/8$, only o_4 is remaining. Hence the first goal is to decide how to define multi-unit-eating *PS* when agents have less than c objects to eat. We resort to the following definition of multi-unit-eating *PS*.

Let $rem(t)$ be the number of objects that have not been completely eaten at time t . In multi-unit-eating *PS*, each agent eats his $\min(c, rem(t))$ most preferred available objects with speed 1 at every time point until all the objects have been consumed.

Agent 1	o_1, o_2	o_1, o_3	o_1, o_4	o_4	o_4	
Agent 2	o_3, o_2	o_3, o_4	o_1, o_4	o_4	o_4	
	0	1/2	3/4	7/8	1	9/8

$$1 : o_1, o_2, o_3, o_4$$

$$2 : o_3, o_2, o_4, o_1$$

$$p = \begin{pmatrix} 3/4 & 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 3/4 & 3/4 \end{pmatrix}.$$

Fig. 1 Illustration of multi-unit-eating *PS* with agents eating their preferred objects over time. The eventual assignment is p .

We will use *MPS* as the abbreviation for multi-unit-eating *PS*. Our first observation is that even though agent may not necessarily eat c objects at each point, each agent eats the same number of objects.

Observation 1 *At each time point, each agent is consuming the same number of objects. All the agents stop eating at exactly the same time.*

If the number of objects is less than c , then we know that only $c' < c$ objects are remaining. Next, we study properties of multi-unit-eating *PS*. The first things to observe is that multi-unit-eating *PS* runs in linear time and results in a unique fractional assignment. We examine various axiomatic properties of multi-unit-eating *PS*. Our main findings are summarized in the following theorem. We will prove these properties in a series of propositions.

Theorem 3 *Multi-unit-eating *PS* is linear-time, *SD* envy-free, weak *SD*-strategyproof, and unanimous but not ex post efficient.*

4.1 Fairness

We first show that multi-unit-eating *PS* satisfies all the notions of fairness defined in the preliminaries. It is easy to see that multi-unit-eating *PS* is anonymous and neutral. Next we show that multi-unit-eating *PS* is *SD* envy-free. For the proof, we use an extra bit of notation. For each set $S \subseteq O$, let the characteristic vector of S be $\hat{S} = (x_1, \dots, x_m)$ where $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$.

Proposition 1 *Multi-unit-eating *PS* is *SD* envy-free.*

Proof When multi-unit-eating *PS* is run, if at least one of the c most preferred available objects of some agent $i \in N$ is finished, agent i starts eating the next most preferred c available objects. Also note that when an agent cannot consume more units of an object, then no agent can consume more units of the object either. We will refer to such a time-point as a breakpoint. The breakpoints are t_1, \dots, t_l . Let p^k be the partial assignment at breakpoint t_k . We prove by induction over k , the number of breakpoints in the algorithm, that for each agent $i \in N$, his partial allocation $p^k(i) \succeq_i^{SD} p^k(j)$ for all $j \in N$.

For the base case $k = 1$, we know that $p^1(i) \succeq_i^{SD} p^1(j)$ for all $j \in N$ since each agent i was consuming his most preferred c objects. Now let us assume that $p^k(i) \succeq_i^{SD} p^k(j)$. We show that $p^{k+1}(i) \succeq_i^{SD} p^{k+1}(j)$. At time t^k , let the number of objects that have not been completely even be $c' \leq c$. Let us consider the time point $t_k + \delta$ for some arbitrarily small $\delta > 0$. From time point t_k to $t_k + \delta$ each agent i consumes δ amount of c' most preferred objects of $S \subset O$ for which δ amount is still available. Thus $p^k(i)$ is changed to $p^k(i) + \delta(\hat{S})$. In the meanwhile for each j , $p^k(j)$ is changed to $p^k(j) + \delta(\hat{S}')$ where S' consists of c' most preferred objects for which δ amount is still available. Hence, $p^{k+1}(i) \succeq_i^{SD} p^{k+1}(j)$ for each $i, j \in N$. \square

Corollary 1 *Multi-unit-eating *PS* is weak *SD* envy-free. Moreover, for the assignment problem without multi-unit demands, *PS* is *SD* envy-free.*

4.2 Strategyproofness

In this subsection, we examine the strategic aspects of multi-unit-eating PS . We show that multi-unit-eating PS satisfies DL -strategyproofness and hence weak SD -strategyproofness. A random assignment function f is DL -strategyproof if $f(\succsim_i)(i) \succsim_i^{DL} f(\succsim'_i, \succsim_{-i})(i)$ for all $\succsim'_i \in \mathcal{R}(O)$ and $\succsim_{-i} \in \mathcal{R}(O)^{n-1}$.

Lemma 1 *DL -strategyproofness implies weak SD -strategyproofness.*

Next we show that multi-unit-eating PS is DL -strategyproof. The key to our argument is the insight that an agent cannot get an object with probability one if he does not start eating it from time $t = 0$. This contrasts sharply with one-at-a-time PS where an agent can still get an object completely even if he delays eating it.

Lemma 2 *An agent cannot get an object o completely if he does not express it as one of his most preferred c objects.*

Proof Assume that agent i does not report o as one of his most preferred c objects but gets it completely. Then while i is eating o , there must be at least $c+1$ objects that are still not eaten completely and none of the other agents are eating o . Before agent i eats o , the number of units eaten by i is at least 1 and less than c . If i has already eaten exactly c units, then it will get zero units of o . Now for the c objects it starts eating including o , it can eat at most $c - 1$ units because it has already eaten at least one unit. Therefore, agent i can eat at most $(c - 1)/c$ of o . \square

Proposition 2 *Multi-unit-eating PS is DL -strategyproof.*

Proof We show that for each agent $i \in N$, $MPS(N, O, (\succsim_i, \succsim_{-i}))(i) \succsim_i^{DL} MPS(N, O, (\succsim'_i, \succsim_{-i}))(i)$ for all other preferences $\succsim'_i \in \mathcal{R}(O)$ and $\succsim_{-i} \in \mathcal{R}(O)^{n-1}$. If agent i misreports but eats the same objects at each time point, then i gets exactly the same allocation. Therefore, it is sufficient to show that i gets a less preferred allocation with respect to DL if he does not eat the most preferred available objects at each point. Consider the untruthful report \succsim'_i under which at some breakpoint t , agent i eats a different set of $\min(c, \text{rem}(t))$ objects than when he reports \succsim_i . Consider the most preferred object o that i started eating at time t when he reports \succsim_i but does not eat when he reports \succsim'_i . This means that for all $o' \succ_i o$, agent i gets exactly the same units of o' when he reports \succsim_i or when he reports \succsim'_i . Since i does not eat o at time t when he reports \succsim'_i , he eats it at a time later than t . We can assume that $\text{rem}(t) > c$ or else agent i will eat the same objects after time t whether he reports \succsim_i or \succsim'_i . We show that i gets strictly less fraction of o when he reports \succsim'_i . We distinguish between two cases: (1) when i eats o when he reports \succsim'_i , there is at least one other agent j that also eats o at some point. (2) when i eats o when he reports \succsim'_i , there is at least one other agent j that also eats o at some point. In case of (1), o' is in demand and i could have eaten a bigger portion of o had he started eating it earlier such as time t . In case of (2), no agent started eating o at any time point when i reports \succsim'_i . This implies that i gets o completely. But this is a contradiction because we proved in Lemma 2 that if an agent does not start eating an object at time 0, then he cannot eat it completely. \square

The proposition implies that Multi-unit-eating PS is weak SD -SP. As a corollary we also get that for $m = n$, the original PS is weak SD -strategyproof. Our proof simplifies the argument in (Step 2, Proposition 1, Bogomolnaia and Moulin, 2001).

Note that Proposition 2 crucially depends on the fact that in MPS, each agent tries to eat his c most preferred objects. If each agent eats $c - 1$ most preferred objects, then we already know from (Kojima, 2009), that the rule is then not even weak SD -strategyproof. We note that in contrast to Multi-unit-eating PS , OPS is not DL-strategyproof and in fact there exists a polynomial-time algorithm for computing a DL best response (Aziz et al., 2015).

4.3 Efficiency

We now consider efficiency of multi-unit-eating PS . We first observe that multi-unit-eating PS satisfies unanimity.

Proposition 3 *Multi-unit-eating PS satisfies unanimity.*

Proof A preference profile admits a perfect assignment only if each agent can get his most preferred c objects. This implies that for any two agents, their sets of c most preferred objects don't intersect. Given this condition, multi-unit-eating PS will assign each agent with his most preferred c objects. \square

Although unanimity is a very undemanding efficiency property, not all assignment rules satisfy unanimity. For example, the uniform rule does not satisfy it. Even if multi-unit-eating PS is modified slightly so that agents eat their $c + 1$ most preferred objects at the same rate, then the modified rule would not satisfy unanimity. We also note that the allocation of each agent via multi-unit-eating PS is SD -preferred over the uniform allocation.

Proposition 4 *For each agent $i \in N$, i SD -prefers his allocation returned by multi-unit-eating PS to the uniform allocation.*

Informally, an agent gets his worst possible assignment if all the other agents have the same preferences. Even in this case, each agent gets a uniform allocation. Although, multi-unit-eating PS satisfies unanimity, an assignment returned by multi-unit-eating PS can be represented as a convex combination of Pareto dominated discrete assignments.

Proposition 5 *There exists a preference profile for which the outcome of multi-unit-eating PS can be represented as a probability distribution over Pareto dominated discrete assignments.*

Proof Consider two agents having the following preferences.

$$1 : o_1, o_2, o_3, o_4$$

$$2 : o_2, o_1, o_4, o_3$$

The random assignment as a result of multi-unit-eating PS is

$$\begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}$$

which can be represented by a probability distribution over the following discrete assignments.

$$\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

It can be shown that both discrete assignments are not SD -efficient. \square

Corollary 2 *Multi-unit-eating PS is not SD -efficient.*

Proof An SD -efficient assignment cannot be represented as a convex combination of discrete assignment in which at least one of the assignments is not SD -efficient. If this were the case, then the random assignment is not SD -efficient. \square

Although the lack of SD -efficiency of multi-unit-eating PS was commented on in the original paper of Che and Kojima (2010), we show that multi-unit-eating PS is surprisingly not even ex post efficient.

Proposition 6 *Multi-unit-eating PS is not ex post efficient even if we allow convex combinations of all deterministic assignments including unbalanced deterministic assignments.*

Proof Consider two agents having the following preferences.

$$\begin{aligned} 1 &: o_1, o_2, o_3, o_4 \\ 2 &: o_3, o_2, o_4, o_1 \end{aligned}$$

A discrete assignment is not SD -efficient if agent 1 gets o_3 or o_4 and agent 2 gets o_1 . The only SD -efficient discrete assignments are $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. We note that the outcome of multi-unit-eating PS is $p = \begin{pmatrix} 7/8 & 4/8 & 2/8 & 3/8 \\ 1/8 & 4/8 & 6/8 & 5/8 \end{pmatrix}$. Now if random assignment p is ex post efficient, then it can be expressed as a convex combination of SD -efficient feasible discrete assignments. Since $p(2)(o_1) > 0$, this is only possible if $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ is used in the convex combination. But since agent 2 does not get o_1 in any other discrete permutation, this means that if any convex combination of SD -efficient discrete assignments is used to obtain p , then in each discrete SD -efficient assignment used the following three cases can occur: (i) 2 gets both o_2 and o_1 ; (ii) 2 gets neither o_2 nor o_1 and (iii) 2 gets o_2 but not o_1 . Hence, it must be that $p(2)(o_2) \geq p(2)(o_1)$. But this is a contradiction. \square

	Uniform	Priority	RP	OPS	MPS
SD -efficiency	-	+	-	+	-
ex post efficient	-	+	+	+	-
unanimity	-	+	+	+	+
SD envy-freeness	+	-	-	+	+
weak SD envy-freeness	+	-	+	+	+
anonymous	+	-	+	+	+
neutrality	+	+	+	+	+
SD -SP	+	+	+	-	-
DL -SP	+	+	+	-	+
weak SD -SP	+	+	+	-	+
polynomial-time	+	+	-	+	+

Table 1 Assignment rules for allocating multiple objects to agents with strict preferences. Most of the properties of rules other than MPS are stated in Kojima (2009).

5 Conclusions

In this paper, we showed a general impossibility result concerning randomized assignment with multi-unit demands. Another impossibility result requiring weak SD -group-strategyproofness applies to randomized assignment without multi-unit demands. As a corollary of the second impossibility, we also obtain the corresponding impossibility in the domain of randomized voting.

We then presented a definition of multi-unit-eating PS . Multi-unit-eating PS has previously only been defined inaccurately in the literature. We showed that whereas multi-unit-eating PS satisfies some compelling fairness and strategic properties, it does not satisfy reasonable efficiency requirements. We note that the positive results of Multi-unit-eating PS even hold if m is not a multiple of n . In this case, agents eat a maximum of $\lceil m/n \rceil$ houses at any time.

Our findings concerning multi-unit-eating PS are summarized in Table 1 which also provides a comparison with other random assignment rules. In view of the impossibility result (Theorem 1), it is not possible to achieve the desirable properties of PS and multi-unit-eating PS simultaneously. It is easy to see that the choice of an assignment rule depends on which properties are prioritized. Our paper helps clarify the relative merits of various randomized assignments rules. It is an open problem whether ex post efficiency, weak SD -strategyproofness and SD envy-freeness are compatible in the multi-unit case. We leave a characterization of multi-unit-eating PS for future work.

Acknowledgments

This material is based upon work supported by the Australian Government’s Department of Broadband, Communications and the Digital Economy, the Australian Research Council, the Asian Office of Aerospace Research and Development through grant AOARD-124056.

References

- A. Abdulkadiroğlu and T. Sönmez. Random serial dictatorship and the core from random endowments in house allocation problems. *Econometrica*, 66(3):689–702, 1998.
- A. Abdulkadiroğlu and T. Sönmez. House allocation with existing tenants. *Journal of Economic Theory*, 88(2):233–260, 1999.
- D. J. Abraham, K. Cechlárová, D. Manlove, and K. Mehlhorn. Pareto optimality in house allocation problems. In *Proceedings of the 16th International Symposium on Algorithms and Computation (ISAAC)*, volume 3341 of *Lecture Notes in Computer Science (LNCS)*, pages 1163–1175, 2005.
- S. Athanassoglou and J. Sethuraman. House allocation with fractional endowments. *International Journal of Game Theory*, 40(3):481–513, 2011.
- H. Aziz and P. Stursberg. A generalization of probabilistic serial to randomized social choice. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI)*, pages 559–565. AAAI Press, 2014.
- H. Aziz, F. Brandt, and M. Brill. The computational complexity of random serial dictatorship. *Economics Letters*, 121(3):341–345, 2013a.
- H. Aziz, F. Brandt, and M. Brill. On the tradeoff between economic efficiency and strategyproofness in randomized social choice. In *Proceedings of the 12th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 455–462. IFAAMAS, 2013b.
- H. Aziz, F. Brandt, and P. Stursberg. On popular random assignments. In *Proceedings of the 6th International Symposium on Algorithmic Game Theory (SAGT)*, volume 8146 of *Lecture Notes in Computer Science (LNCS)*, pages 183–194. Springer-Verlag, 2013c.
- H. Aziz, S. Gaspers, S. Mackenzie, N. Mattei, N. Narodytska, and T. Walsh. Manipulating the probabilistic serial rule. In *Proceedings of the 14th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 1451–1459, 2015.
- A. Bhalgat, D. Chakrabarty, and S. Khanna. Social welfare in one-sided matching markets without money. In *Proceedings of APPROX-RANDOM*, pages 87–98, 2011.
- A. Bogomolnaia and H. Moulin. A new solution to the random assignment problem. *Journal of Economic Theory*, 100(2):295–328, 2001.
- S. Bouveret and J. Lang. A general elicitation-free protocol for allocating indivisible goods. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, pages 73–78. AAAI Press, 2011.
- S. Bouveret, U. Endriss, and J. Lang. Fair division under ordinal preferences: Computing envy-free allocations of indivisible goods. In *Proceedings of the 19th European Conference on Artificial Intelligence (ECAI)*, pages 387–392, 2010.
- E. Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.
- E. Budish, Y.-K. Che, F. Kojima, and P. Milgrom. Designing random allocation mechanisms: Theory and applications. *American Economic Review*, 103(2):585–623,

- 2013.
- C. Chambers. Consistency in the probabilistic assignment model. *Journal of Mathematical Economics*, 40:953–962, 2004.
- Y.-K. Che and F. Kojima. Asymptotic equivalence of probabilistic serial and random priority mechanisms. *Econometrica*, 78(5):1625–1672, 2010.
- H. Crès and H. Moulin. Scheduling with opting out: Improving upon random priority. *Operations Research*, 49(4):565–577, 2001.
- L. Ehlers and B. Klaus. Probabilistic assignments of identical indivisible objects and probabilistic uniform rules. *Review of Economic Design*, 8:249–268, 2003.
- P. Gärdenfors. Assignment problem based on ordinal preferences. *Management Science*, 20:331–340, 1973.
- M. Guo and V. Conitzer. Strategy-proof allocation of multiple items without payments or priors. In *Proceedings of the 9th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 881–888. IFAAMAS, 2010.
- T. Hashimoto. The generalized random priority mechanism with budgets. 2013.
- J. W. Hatfield. Strategy-proof, efficient, and nonbossy quota allocations. *Social Choice and Welfare*, 33:505–515, 2009.
- E. J. Heo. Probabilistic assignment with multiple demands: A generalization and a characterization of the serial rule. Technical Report 1809195, SSRN, 2011.
- A. Hylland and R. Zeckhauser. The efficient allocation of individuals to positions. *The Journal of Political Economy*, 87(2):293–314, 1979.
- T. Kalinowski, N. Narodytska, T. Walsh, and L. Xia. Strategic behavior when allocating indivisible goods sequentially. In *Proceedings of the 27th AAAI Conference on Artificial Intelligence (AAAI)*, pages 452–458. AAAI Press, 2013.
- A.-K. Katta and J. Sethuraman. A solution to the random assignment problem on the full preference domain. *Journal of Economic Theory*, 131(1):231–250, 2006.
- F. Kojima. Random assignment of multiple indivisible objects. *Mathematical Social Sciences*, 57(1):134–142, 2009.
- T. Nguyen, A. Peivandi, and R. Vohra. Assignment problems with complementarities. 2015.
- L.-G. Svensson. Queue allocation of indivisible goods. *Social Choice and Welfare*, 11:323–330, 1994.
- L-G Svensson. Strategy-proof allocation of indivisible goods. *Social Choice and Welfare*, 16(4):557–567, 1999.
- L. Wilson. Assignment using choice lists. *Operations Research Quarterly*, 28(3):569–578, 1977.
- O. Yilmaz. Random assignment under weak preferences. *Games and Economic Behavior*, 66(1):546–558, 2009.
- H. P. Young. Dividing the indivisible. *American Behavioral Scientist*, 38:904–920, 1995.